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Chaotic semigroups generated by certain differential operators of order 1

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Abstract. We consider the initial value problem of a partial differential equation $\dfrac{\partial u}{\partial t} = c(x)\dfrac{\partial u}{\partial x} + g(x,u)$ in some function spaces X on the interval I of the real line. By using the representation formula of the solution to the equation, we define a C_0-semigroup $\{T_t\}_{t\geq 0}$ of bounded linear operators on X. When $c(x) = \gamma x$ ($\gamma \in \mathbb{R}$), $g(x,u) = h(x)u$ ($h \in C(I,\mathbb{C})$) and I is [0,1] or $[1,\infty)$, we give sufficient conditions for the semigroup to be chaotic by using the spectral property of its infinitesimal generator. When $c(x) = 1$ and $g(x,u) = h(x)u$, we also give sufficient conditions for the semigroup to be chaotic by using the property of an admissible weight function.

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§1. Introduction

The equation

$$\frac{\partial u}{\partial t} = c(x)\frac{\partial u}{\partial x} + g(x,u) \qquad (x, t \geq 0) \tag{1.1}$$

has been used to model the dynamics of a population of cells undergoing simultaneous proliferation and maturation, where x is the maturation variable ([4], [5]). The solution of (1.1) has some connection with Wiener process. In fact, A. Lasota and M. C. Mackey [3] showed how to construct an exact, continuous time, semidynamical system that corresponds to the partial differential equation above with $c(x) = -x$ ($x \in [0,1]$) and $g(x,u) = \frac{1}{2}u$ by using a one-dimensional Wiener process.

In this paper, we consider some special cases of (1.1) which generates chaotic semigroups and generalize the result of A. Lasota and M. C. Mackey [3].

In §2, we consider the space $X_1 = \{f \in C([0,1],\mathbb{C}) \mid f(0) = 0\}$ and the following initial value problem of the partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma x \dfrac{\partial u}{\partial x} + h(x)u, \\ u(0,x) = f(x), \end{cases} \tag{1.2}$$

where $\gamma \in \mathbb{C}$, $h \in C([0,1],\mathbb{C})$ and $f \in X_1$.

If $u(t,x)$ is the classical solution of (1.2) for $f \in C^1([0,1],\mathbb{C}) \cap X_1$, then it must be of the form $u(t,x) = f(e^{\gamma t}x)\exp(\int_0^t h(e^{\gamma(t-s)}x)\,ds)$. In order to satisfy $e^{\gamma t}x \in [0,1]$ for $x \in [0,1]$ and $t \geq 0$, γ must be a non-positive number. In this case, by using this representation formula $f(e^{\gamma t}x)\exp(\int_0^t h(e^{\gamma(t-s)}x)\,ds)$ of the solution of (1.2), we can define the bounded linear operators $\{T_t\}_{t\geq 0}$ on X_1 by $T_t f(x) = f(e^{\gamma t}x)\exp(\int_0^t h(e^{\gamma(t-s)}x)\,ds)$ for $f \in X_1$. Then $\{T_t\}_{t\geq 0}$ is a strongly continuous semigroup on X_1 (Theorem 1). In this paper we call $\{T_t\}_{t\geq 0}$ *the solution semigroup* on X_1 to the partial differential equation (1.2).

In [1], W. Desch, W. Schappacher and G. F. Webb gave a sufficient condition (Theorem A) for $\{T_t\}_{t\geq 0}$ to be chaotic, by using the eigenvectors of the infinitesimal generator A of the strongly continuous semigroup $\{T_t\}_{t\geq 0}$. By applying their result to the solution semigroup, we give a sufficient condition for the solution semigroup to be chaotic on X_1 (Theorem 1). In §3, we also give a sufficient condition for the solution semigroup to be chaotic on $L^2([0,1],\mathbb{C})$ (Theorem 2).

In §4, we deal with the partial differential equation

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial u}{\partial x} + h(x)u \qquad (x, t \geq 0) \tag{1.3$'$}$$

with the initial condition $u(0,x) = f(x)$ with some $f \in C_0(I,\mathbb{C})$, where $I = [0,\infty)$ and $C_0(I,\mathbb{C})$ is the space of all complex-valued continuous functions on I satisfying $\lim_{x\to\infty} f(x) = 0$. If $u(t,x)$ is the classical solution of (1.3′) for $f \in C_0(I,\mathbb{C})$, then it must be of the form $u(t,x) = e^{\int_x^{x+t} h(s)ds} f(x+\gamma t)$. In order to satisfy $x + t \in [0,\infty)$ for $x \in [0,\infty)$ and $t \geq 0$, γ must be a non-negative number. Since the case $\gamma = 0$ is a special case, we shall consider the case $\gamma > 0$. So by replacing γt by t, we shall consider the following partial differential equation instead of (1.3′):

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + h(x)u. \qquad (x, t \geq 0) \tag{1.3}$$

Since the method of the proof in §2 is not applicable in this case, we use the result in [7] and show that the solution semigroup $\{T_t\}_{t\geq 0}$ to the partial differential equation (1.3) is a chaotic, strongly continuous semigrroup on

$C_0(I,\mathbb{C})$ (Theorem 4), if h is a bounded continuous function on I satisfying $\int_0^\infty h(s)ds = \infty$.

The authors would like to express their deep gratitude to Professor Shizuo Miyajima for his valuable comments and useful advices. At first the function $h(x)$ in Theorems 1 and 2 were considered as a constant function. By his suggestion we improve the theorems by using the function $h(x)$ in $C([0,1],\mathbb{C})$.

§2. Chaotic semigroups on $C(I,\mathbb{C})$

Recall that a family $\{T_t\}_{t\geq 0}$ of bounded linear operators on a Banach space X is called *a strongly continuous semigroup* if it satisfies the following conditions: (1) $T_{t+s} = T_t T_s$ for all $t, s \in \mathbb{R}_+$, (2) $T_0 = Id$, and (3) the mapping : $t \mapsto T_t x$ is continuous from $\mathbb{R}_+$ to X for every $x \in X$. A strongly continuous semigroup $\{T_t\}_{t\geq 0}$ is called *hypercyclic* if there exists $x \in X$ such that the set $\{T_t x \mid t \geq 0\}$ is dense in X. The semigroup $\{T_t\}_{t\geq 0}$ is called *chaotic* if $\{T_t\}_{t\geq 0}$ is hypercyclic and the set of periodic points $X_{\mathrm{per}} = \{x \in X \mid \exists t > 0 \ s.t.\ T_t x = x\}$ is dense in X.

As to a sufficient condition for a semigroup to be chaotic, the following theorem is known.

Theorem A ([1]). *Let X be a separable Banach space and let A be the infinitesimal generator of a strongly continuous semigroup $\{T_t\}_{t\geq 0}$ on X. Let U be an open subset of the point spectrum of A, which intersects the imaginary axis, and for each $\lambda \in U$ let x_λ be a nonzero eigenvector, i.e. $Ax_\lambda = \lambda x_\lambda$. For each $\phi \in X^*$ we define a function F_ϕ: $U \to \mathbb{C}$ by $F_\phi(\lambda) = \langle \phi, x_\lambda \rangle$. Assume that for each $\phi \in X^*$ the function F_ϕ is analytic and that F_ϕ does not vanish identically on U unless $\phi = 0$. Then $\{T_t\}_{t\geq 0}$ is chaotic.*

We shall apply this theorem to the semigroup related to the following partial differential equation. We consider the space $X_1 = \{f \in C([0,1],\mathbb{C}) \mid f(0) = 0\}$ and the following initial value problem of a partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma x \dfrac{\partial u}{\partial x} + h(x)u \\ u(0,x) = f(x) \end{cases} \tag{2.1}$$

where $\gamma < 0$, $h \in C([0,1],\mathbb{C})$ and $f \in X_1$. By using the representation formula $\exp\left\{\int_0^t h(e^{\gamma(t-s)}x)\, ds\right\} f(e^{\gamma t}x)$ of the classical solution of (2.1), we define the bounded linear operator $\{T_t\}_{t\geq 0}$ on X_1 as follows:

$$T_t f(x) = \exp\left\{\int_0^t h(e^{\gamma(t-s)}x)\,ds\right\} f(e^{\gamma t}x) \qquad \text{for} \qquad f \in X_1.$$

Note that if $\gamma > 0$ then $e^{\gamma t}x \notin [0,1]$ holds for $x \in (e^{-\gamma t}, 1]$. Since we are interested in the case $\gamma \neq 0$, we suppose $\gamma < 0$. Since the equations $T_{t_1+t_2}f(x) = \exp\left\{\int_0^{t_1+t_2} h(e^{\gamma(t_1+t_2-s)}x)\,ds\right\} f(e^{\gamma(t_1+t_2)}x) = T_{t_1} \cdot T_{t_2}f(x)$ and $T_0 f(x) = f(x)$ hold for any $f \in X_1$, $\{T_t\}_{t\geq 0}$ is a semigroup. Moreover the semigroup $\{T_t\}_{t\geq 0}$ becomes a strongly continuous semigroup on X_1. The proof of continuity is shown in the following theorem. Recall that the infinitesimal generator $A\colon D(A) \subseteq X_1 \to X_1$ of the strongly continuous semigroup $\{T_t\}_{t\geq 0}$ on X_1 is given by

$$Af = \lim_{t\downarrow 0} \frac{T_t f - f}{t}$$

for every f in its domain

$$D(A) = \left\{ f \in X_1 \,\middle|\, \lim_{t\downarrow 0} \frac{T_t f - f}{t} \text{ exists.} \right\}.$$

In this paper we call $\{T_t\}_{t\geq 0}$ *the solution semigroup* to the partial differential equation. By applying Theorem A to the solution semigroup, we have a sufficient condition for the solution semigroup to be chaotic.

Theorem 1. *Let X_1 be the space $\{f \in C([0,1],\mathbb{C}) \mid f(0) = 0\}$ with sup norm. We consider the following initial value problem of a partial differential equation:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma x \dfrac{\partial u}{\partial x} + h(x)u \\ u(0,x) = f(x) \end{cases} \tag{2.2}$$

where $\gamma < 0$, $h \in C([0,1],\mathbb{C})$ and $f \in X_1$. Then the solution semigroup $\{T_t\}_{t\geq 0}$ ($T_t f(x) = \exp\left\{\int_0^t h(e^{\gamma(t-s)}x)\,ds\right\} f(e^{\gamma t}x)$) to the partial differential equation is a strongly continuous semigroup on X_1. Moreover if $\min\{\Re(h(x)) \mid x \in [0,1]\}$ is positive, then $\{T_t\}_{t\geq 0}$ is chaotic.

Proof. Put $a = \sup_{0\ x\ 1} |h(x)|$. For $f \in X_1$, we have

$$\begin{aligned} \|T_t f - f\| &= \sup_{0\ x\ 1} |e^{\int_0^t h(e^{\gamma(t-s)}x)ds} f(e^{\gamma t}x) - f(x)| \\ &\cdot\ |e^{at} - 1| \sup_{0\ x\ 1} |f(e^{\gamma t}x)| + \sup_{0\ x\ 1} |f(e^{\gamma t}x) - f(x)| \\ &= |e^{at} - 1| \|f\| + \sup_{0\ x\ 1} |f(e^{\gamma t}x) - f(x)|, \end{aligned}$$

which implies the strong continuity of $\{T_t\}_{t\geq 0}$.

We shall show that $\{T_t\}_{t\geq 0}$ is chaotic if $\min\{\Re(h(x)) \mid x \in [0,1]\} > 0$. To show that all assumptions of Theorem A hold, we verify the following:
(i) X_1 is a separable Banach space.
(ii) The existence of an open set U of the point spectrum of the infinitesimal generator A which intersects the imaginary axis.
(iii) For $\lambda \in U$, put $f_\lambda(x) = \exp(-\frac{1}{\gamma}\int_x^1 \frac{\lambda - h(s)}{s} ds)$. For each $\phi \in X_1^*$ we define a function $F_\phi : U \to \mathbb{C}$ by $F_\phi(\lambda) = \langle \phi, f_\lambda \rangle$. Then for each $\phi \in X_1^*$ the function F_ϕ is analytic on U.
(iv) If $F_\phi = 0$ on U, then $\phi = 0$.

(i) It is clear that X_1 is a separable Banach space by Weierstrass approximation theorem.
(ii) Let $A \colon D(A) \subseteq X_1 \to X_1$ be the infinitesimal generator of the strongly continuous semigroup $\{T_t\}_{t\geq 0}$. Put

$$D_1 = \left\{ f \in X_1 \cap C^1((0,1], \mathbb{C}) \;\middle|\; \lim_{x\to 0} x f'(x) = 0 \right\}.$$

Then we shall show that $D_1 = D(A)$ holds. For $f \in D(A)$, Af belongs to X_1 and f is differentiable on (0,1). By a standard argument, we can see that $Af(x) = h(x)f(x) + \gamma x f'(x)$ holds for $x \in (0,1]$. So $\lim_{x\to 0} x f'(x) = 0$, which implies $D(A) \subset D_1$. Conversely, suppose $f \in D_1$. Then $hf + \gamma x f' \in X_1$. So for any $\varepsilon > 0$, there exists $1 > \delta_1 > 0$ such that $|h(x)f(x) + \gamma x f'(x)| < \varepsilon$ for any $x \in [0, \delta_1]$, $|xf'(x) - x'f'(x')| < \varepsilon$ and $|f(x) - f(x')| < \varepsilon$ for any $x, x' \in [0,1]$ with $|x - x'| < \delta_1$. Since h is continuous, there exists $\delta_2 > 0$ such that $|h(e^{\gamma s}x) - h(x)| < \varepsilon$ for every $0 \leq s < \delta_2$ and $x \in [0,1]$. So we have

$$\begin{aligned}
\left| \frac{e^{\int_0^t h(e^{\gamma(t-s)}x)ds} - 1}{t} - h(x) \right| &< \left| \frac{\int_0^t h(e^{\gamma(t-s)}x)ds}{t} - h(x) \right| + \frac{t}{2}||h||_\infty^2 e^{t||h||_\infty} \\
&< \varepsilon + 2t||h||_\infty^2 < 2\varepsilon
\end{aligned}$$

for $0 \leq t < \delta_3$, where $\delta_3 = \min\left\{\delta_2, \frac{1}{||h||_\infty}, \frac{\varepsilon}{2||h||_\infty^2}\right\}$. For $0 < t < \min\{\frac{1}{\gamma}\log(1 - \delta_1), \delta_3\}$, by using the relations $0 \leq x - e^{\gamma t}x < \delta_1$ and $f(e^{\gamma t}x) - f(x) =$

$\int_0^t \gamma e^{\gamma s} x f'(e^{\gamma s}x)ds$, we have

$$\begin{aligned}
&\left|\frac{T_t f(x) - f(x)}{t} - (\gamma x f'(x) + h(x)f(x))\right| \\
\cdot\quad &\left|\frac{e^{\int_0^t h(e^{\gamma(t-s)}x)ds} - 1}{t} f(e^{\gamma t}x) - h(x)f(x)\right| \\
&+\frac{1}{t}\int_0^t |\gamma e^{\gamma s} x f'(e^{\gamma s}x) - \gamma x f'(x)|ds \\
\cdot\quad &(2||f||_\infty + ||h||_\infty + \gamma)\varepsilon,
\end{aligned}$$

which implies $D_1 \subset D(A)$. Hence $D(A) = D_1$.

Put $\alpha = \min\{\Re(h(x)) \mid x \in [0,1]\}$ and

$$U = \{\lambda \in \mathbb{C} \mid \Re(\lambda) < \alpha\}.$$

Since we assumed $\alpha > 0$, the set U intersects the imaginary axis. For $\lambda \in U$, $f_\lambda(x) = \exp(-\frac{1}{\gamma}\int_x^1 \frac{\lambda - h(s)}{s}ds)$ is continuous on [0,1]. It is easy to see that $f_\lambda(x)$ belongs to $D_1 = D(A)$ and satisfies $Af_\lambda = \lambda f_\lambda$. So U is an open subset of the point spectrum of A.

(iii) Let $\lambda \in U$. Put $v_{p,\lambda}(x) = \dfrac{f_{\lambda+p}(x) - f_\lambda(x)}{p}$ for $p \neq 0$ with $|p|$ small enough and set $g_\lambda(x) = \frac{\log x}{\gamma}\exp(-\frac{1}{\gamma}\int_x^1 \frac{\lambda - h(s)}{s}ds)$ for $x \in (0,1]$ and $g_\lambda(0) = 0$. Since $\lim_{x\to 0} g_\lambda(x) = 0$, we have $g_\lambda \in X_1$. By using the relation $f_{\lambda+p}(x) - f_\lambda(x) = p\int_0^1 \frac{\log x}{\gamma}\exp(-\frac{1}{\gamma}\int_x^1 \frac{\lambda - h(s) + tp}{s}ds)dt$, we have for $x \in (0,1]$,

$$\begin{aligned}
&v_{p,\lambda}(x) - g_\lambda(x) \\
= \quad &\int_0^1 \frac{\log x}{\gamma}\left\{\exp(-\frac{1}{\gamma}\int_x^1 \frac{\lambda - h(s) + tp}{s}ds) - \exp(-\frac{1}{\gamma}\int_x^1 \frac{\lambda - h(s)}{s}ds)\right\}dt \\
= \quad &g_\lambda(x)\int_0^1 (x^{\frac{tp}{\gamma}} - 1)dt.
\end{aligned}$$

Put $c = \frac{\alpha - \Re(\lambda)}{2} > 0$. For any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $|\frac{\log x}{\gamma}\exp(-\frac{1}{\gamma}\int_x^1 \frac{\lambda - h(s) + c}{s}ds)| < \varepsilon$ for $0 \cdot\ x < \delta_1$, and there exists $\delta_2 > 0$ such that $|x^{\frac{tp}{\gamma}} - 1| < \frac{\varepsilon}{\|g_\lambda\|}$ for $\delta_1 \cdot\ x \cdot\ 1$ and $0 < |p| < \delta_2$.

For $x \in [0, \delta_1]$ and $0 < |p| < c$, we have

$$\begin{aligned}
&|v_{p,\lambda}(x) - g_\lambda(x)| \\
\cdot\quad &\int_0^1 \left\{\left|\frac{\log x}{\gamma}\exp(-\frac{1}{\gamma}\int_x^1 \frac{\lambda - h(s) + tp}{s}ds)\right|\right. \\
&\left.+\left|\frac{\log x}{\gamma}\exp(-\frac{1}{\gamma}\int_x^1 \frac{\lambda - h(s)}{s}ds)\right|\right\}dt \\
< \quad &2\varepsilon.
\end{aligned}$$

For $x \in [\delta_1, 1]$ and for $0 < |p| < \delta_2$, we have

$$|v_{p,\lambda}(x) - g_\lambda(x)| \cdot \quad |g_\lambda(x)| \int_0^1 |x^{\frac{tp}{\gamma}} - 1| dt < \varepsilon.$$

Hence we have $|v_{p,\lambda}(x) - g_\lambda(x)| < 2\varepsilon$ for $0 < |p| < \min\{c, \delta_2\}$ and for $x \in [0,1]$. So $|v_{p,\lambda}(x) - g_\lambda(x)|$ goes to 0 uniformly on [0,1] as $p \to 0$ and

$$\langle \phi, g_\lambda \rangle = \lim_{p \to 0} \langle \phi, \frac{f_{\lambda+p} - f_\lambda}{p} \rangle = \frac{dF_\phi}{d\lambda}.$$

Therefore $F_\phi(\lambda)$ is analytic with respect to $\lambda \in U$.
(iv) We shall show that if $F_\phi(\lambda) = 0$ for all $\lambda \in U$ then $\phi = 0$. We recall the following: $U = \{\lambda \in \mathbb{C} \mid \Re(\lambda) < \alpha\}$ and $f_\lambda(x) = \exp(-\frac{1}{\gamma}\int_x^1 \frac{\lambda - h(s)}{s} ds)$ for $\lambda \in U$. Take a real constant λ_0 satisfying $\lambda_0 < \alpha$. For $\Re(\lambda) < \lambda_0$,

$$\begin{aligned} f_\lambda(x) &= \exp\left\{-\frac{1}{\gamma}\int_x^1 \frac{\lambda - \lambda_0}{s} ds - \frac{1}{\gamma}\int_x^1 \frac{\lambda_0 - h(s)}{s} ds\right\} \\ &= \exp\left\{-\frac{1}{\gamma}\int_x^1 \frac{\lambda - \lambda_0}{s} ds\right\} \exp\left\{-\frac{1}{\gamma}\int_x^1 \frac{\lambda_0 - h(s)}{s} ds\right\} \qquad (2.3) \end{aligned}$$

holds. Put the second factor of (2.3) as follows:

$$q(x) = \exp\left\{-\frac{1}{\gamma}\int_x^1 \frac{\lambda_0 - h(s)}{s} ds\right\}.$$

It is easy to see that q is continuous on $[0,1]$ and positive except for $x = 0$. The first factor of (2.3) becomes

$$\exp\left\{-\frac{\lambda - \lambda_0}{\gamma} \cdot \log(\frac{1}{x})\right\} = x^{\frac{\lambda - \lambda_0}{\gamma}}.$$

For $n \in \{1, 2, 3, \cdots\}$, put $\lambda_n = \gamma n + \lambda_0$. The assumption $\gamma < 0$ implies $\lambda_n \in U$ for $n = 1, 2, 3, \cdots$.

Then we have $f_{\lambda_n}(x) = x^n q(x)$ for $n = 1, 2, 3, \cdots$.

¿From the assumption, $0 = F_\phi(\lambda_n) = \langle \phi, f_{\lambda_n} \rangle = \langle \phi, x^n q \rangle$ holds for $n = 1, 2, 3, \cdots$. By the Stone-Weierstrass theorem, the linear span of $\{x^n \mid n = 1, 2, 3, \cdots\}$ is dense in X_1. Since $q(x) > 0$ for any $x \in (0,1]$, the linear span of $\{x^n q \mid n = 1, 2, 3, \cdots\}$ is also dense in X_1. So we have $\phi = 0$.

By (i) to (iv), all assumptions of Theorem A hold. So $\{T_t\}_{t \geq 0}$ is chaotic by Theorem A. □

The space $Y_1 = \{f \in C([1,\infty), \mathbb{C}) \mid \lim_{x \to \infty} f(x) = 0\}$ has relation with the space $X_1 = \{f \in C([0,1], \mathbb{C}) \mid f(0) = 0\}$ by the mapping $\phi : X_1 \to Y_1$ defined by $(\phi f)(x) = f(\frac{1}{x})$. So we shall consider the corresponding equation in Y_1 to the equation (2.2) considered in X_1 as follows:

$$\frac{\partial u}{\partial t} = -\gamma y \frac{\partial u}{\partial y} + h(y)u. \tag{2.4}$$

Let $\{T_t\}_{t\geq 0}$ be the solution semigroup on X_1 with respect to (2.2) and $\{S_t\}_{t\geq 0}$ be the solution semigroup on Y_1 generated from the classical solution of (2.4). Then the following diagram commutes.

$$\begin{array}{ccc} X_1 & \xrightarrow{T_t} & X_1 \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y_1 & \xrightarrow[S_t]{} & Y_1 \end{array}$$

Hence we have the following.

Corollary.
Let Y_1 be the space $\{f \in C([1,\infty),\mathbb{C}) \mid \lim_{x\to\infty} f(x) = 0\}$ with sup norm. We consider the following initial value problem of a partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma x \dfrac{\partial u}{\partial x} + h(x)u \\ u(0,x) = f(x) \end{cases}$$

where $\gamma > 0$, $f \in Y_1$, $h \in C([1,\infty),\mathbb{C})$ and $\lim_{x\to\infty} h(x)$ exists. Then the solution semigroup $\{S_t\}_{t\geq 0}$ $(S_t f(x) = e^{\int_0^t h(e^{\gamma(t-s)}x)ds} f(e^{\gamma t}x))$ to the partial differential equation is a strongly continuous semigroup on Y_1.

Moreover if $\inf\{\Re h(x) \mid x \in [1,\infty)\} > 0$, *then $\{S_t\}_{t\geq 0}$ is chaotic.*

§3. Chaotic semigroups on $L^2(I)$

Let X_2 be the space $L^2([0,1],\mathbb{C})$. We shall consider the partial differential equation in $L^2([0,1],\mathbb{C})$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma x \dfrac{\partial u}{\partial x} + h(x)u \\ u(0,x) = f(x) \end{cases} \tag{3.1}$$

where $\gamma < 0$, $h \in C([0,1],\mathbb{C})$ and $f \in X_2$. By using the representation formula $\exp\left\{\int_0^t h(e^{\gamma(t-s)}x)\,ds\right\} f(e^{\gamma t}x)$ of the classical solution of (3.1), we can define a family $\{T_t\}_{t\geq 0}$ of bounded linear operators on X_2 by $T_t f(x) = \exp\left\{\int_0^t h(e^{\gamma(t-s)}x)\,ds\right\} f(e^{\gamma t}x)$ for $f \in X_2$. Then $\{T_t\}_{t\geq 0}$ is a semigroup.

Moreover the semigroup $\{T_t\}_{t\geq 0}$ is a strongly continuous semigroup on X_2. The proof of continuity is shown in the following theorem. By applying Theorem A to the solution semigroup $\{T_t\}_{t\geq 0}$, we shall give a sufficient condition for the solution semigroup to be chaotic.

Theorem 2.
Let X_2 be the space $L^2([0,1],\mathbb{C})$. We consider the following initial value problem of a partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma x \dfrac{\partial u}{\partial x} + h(x)u \\ u(0,x) = f(x) \end{cases}$$

where $\gamma < 0$, $h \in C([0,1],\mathbb{C})$ and $f \in X_2$. Then the solution semigroup $\{T_t\}_{t\geq 0}$ ($T_t f(x) = \exp\left\{\int_0^t h(e^{\gamma(t-s)}x)\,ds\right\} f(e^{\gamma t}x)$) to the partial differential equation is a strongly continuous semigroup on X_2.

Moreover if $\min\{\Re(h(x)) \mid x \in [0,1]\} > \frac{\gamma}{2}$, *then $\{T_t\}_{t\geq 0}$ is chaotic.*

Proof. To check the strong continuity of $\{T_t\}_{t\geq 0}$, we shall show the continuity of $\{T_t\}_{t\geq 0}$ at $t = 0$.

Let f be an element of X_2. Then for any $\varepsilon > 0$ there exists a continuous function ξ on $[0,1]$ such that

$$||f - \xi||_{L^2} < \frac{\varepsilon}{6}.$$

Since ξ is continuous, there exists $\delta_1 > 0$ such that

$$||T_t\xi - \xi||_\infty < \frac{\varepsilon}{2}$$

holds with $0 < t < \delta_1$, where $||\cdot||_\infty$ is the sup norm . For $k \in L^2$, we have

$$||T_t k||_{L^2} \le e^{\alpha_0 t}||k||_{L^2},$$

where $\alpha_0 = \max_{0\le x\le 1}\{\Re(h(x))\} - \frac{\gamma}{2}$. Put $\delta = \min(\delta_1, \frac{\log 2}{\alpha_0})$. Then we have

$$\begin{aligned} ||T_t f - f||_{L^2} &\le ||T_t f - T_t\xi||_{L^2} + ||T_t\xi - \xi||_{L^2} + ||\xi - f||_{L^2} \\ &\le e^{\alpha_0 t}||f-\xi||_{L^2} + ||T_t\xi - \xi||_\infty + ||f - \xi||_{L^2} \\ &< ||f-\xi||_{L^2}(1 + e^{\alpha_0 t}) + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{6}(1+2) + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for $t \in (0,\delta)$. So $\{T_t\}_{t\geq 0}$ is a strongly continuous semigroup.

Hereafter we shall check the following (i) – (iv) as in the proof of Theorem 1 to show that all assumptions of Theorem A hold if $\min\{\Re(h(x)) \mid x \in [0,1]\} > \frac{\gamma}{2}$ holds.
(i) X_1 is a separable Banach space.
(ii) The existence of an open set U of the point spectrum of the infinitesimal generator A which intersects the imaginary axis.
(iii) For $\lambda \in U$, put $f_\lambda(x) = \exp(-\frac{1}{\gamma}\int_x^1 \frac{\lambda - h(s)}{s} ds)$. For each $\phi \in X_1^*$ we define a function $F_\phi : U \to \mathbb{C}$ by $F_\phi(\lambda) = \langle \phi, f_\lambda \rangle$. Then for each $\phi \in X_1^*$ the function F_ϕ is analytic on U.
(iv) If $F_\phi = 0$ on U, then $\phi = 0$.

(i) It is obvious.
(ii) Let $A\colon D(A) \subseteq X_1 \to X_1$ be the infinitesimal generator of the strongly continuous semigroup $\{T_t\}_{t\geq 0}$. Put

$$D_2 = \left\{ f \in X_2 \,\middle|\, xf \text{ is absolutely continuous and } (xf)' \in X_2 \right\}.$$

We recall that $f \in D_2$ holds if and only if $f \in X_2$ and xf belongs to the Sobolev space $H^1(0,1)$. For $f \in D(A)$, there exists $g \in X_2$ such that $\lim_{t\downarrow 0} \frac{T_t f - f}{t} = g$. Since f is integrable on [0,1], we see that for $l, m \in [0,1]$

$$\begin{aligned}
&\int_l^m \frac{T_t f(x) - f(x)}{t} dx = \int_l^m \frac{e^{\int_0^t h(e^{\gamma(t-s)}x)ds} f(e^{\gamma t}x) - f(x)}{t} dx \\
= &\int_{le^{\gamma t}}^{me^{\gamma t}} \frac{e^{\int_0^t h(e^{-\gamma s}x)ds - \gamma t}}{t} f(x)dx - \int_l^m \frac{f(x)}{t} dx \\
= &\frac{1}{l - le^{\gamma t}} \int_{le^{\gamma t}}^{l} \frac{l(1-e^{\gamma t})}{t} e^{\int_0^t h(e^{-\gamma s}x)ds - \gamma t} f(x)dx \\
&+ \int_l^m \frac{e^{\int_0^t h(e^{-\gamma s}x)ds - \gamma t} - 1}{t} f(x)dx \\
&- \frac{1}{m - me^{\gamma t}} \int_{me^{\gamma t}}^{m} \frac{m(1-e^{\gamma t})}{t} e^{\int_0^t h(e^{-\gamma s}x)ds - \gamma t} f(x)dx
\end{aligned}$$

converges to

$$-l\gamma f(l) + \int_l^m (h(x) - \gamma) f(x)dx + m\gamma f(m)$$

as $t \downarrow 0$ for almost all l, m ([6], Theorem 9-8 VI]). However, the left hand side converges to $\int_l^m g(x)dx$. By redefining f on a null set we obtain

$$mf(m) = \int_l^m \frac{1}{\gamma}\{g(x) - (h(x) - \gamma)f(x)\}dx + lf(l),$$

which implies that $xf(x)$ is an absolutely continuous function with derivative (almost everywhere) equal to $\frac{1}{\gamma}\{g(x)-(h(x)-\gamma)f(x)\}$ and hence $(xf)'$ belongs to X_2. So $D(A) \subset D_2$.

Conversely for $f \in D_2$, we have

$$\begin{aligned} \frac{T_t f(x)-f(x)}{t} &- (\gamma x f'(x)+h(x)f(x)) \\ &= \left(\frac{e^{\int_0^t h(e^{\gamma(t-s)}x)ds}-1}{t}-h(x)\right) f(e^{\gamma t}x) \qquad (3.2) \\ &+ h(x)(f(e^{\gamma t}x)-f(x))+\left\{\frac{f(e^{\gamma t}x)-f(x)}{t}-\gamma x f'(x)\right\}. \end{aligned}$$

We will show that each term of (3.2) goes to 0 as $t \to 0$. It is obvious that the norm of the first term of (3.2) converges to 0 as $t \to 0$ in a similar way to that in Theorem 1. For each $\varepsilon > 0$ and each $t (t_0 > t \geq 0)$ with some fixed $t_0 > 0$, there exists $\delta_1 > 0$ such that

$$\int_0^{\delta_1} |f(e^{\gamma t}x)-f(x)|^2 dx < \varepsilon.$$

Since xf is absolutely continuous, f is absolutely continuous on $[\delta_1, 1]$ and $\|h(x)(f(e^{\gamma t}x)-f(x))\|$ converges to 0 as $t \to 0$.

Put $\eta(x) = \gamma x f'(x)$. Then $f \in D_2$ implies $\eta \in X_2$. For any $\varepsilon > 0$, there exists $\xi \in C([0,1], \mathbb{C})$ such that $\|\xi - \eta\| < \varepsilon$ and there exists $\delta > 0$ such that $\|\xi(e^{\gamma s}x) - \xi(e^{\gamma t}x)\| < \varepsilon$ for any $0 \le s \le t < \delta$ and any $0 \le x \le 1$. Moreover, for $0 \le s < \delta$,

$$\begin{aligned} \|\eta(e^{\gamma s}x)-\xi(e^{\gamma s}x)\|^2 &= \int_0^1 (\eta(e^{\gamma s}x)-\xi(e^{\gamma s}x))^2 dx \\ &= \int_0^{e^{\gamma s}} (\eta(y)-\xi(y))^2 e^{-\gamma s} dy \le e^{-\gamma\delta}\|\eta-\xi\|^2. \end{aligned}$$

So $\|\eta(e^{\gamma s}x)-\eta(e^{\gamma t}x)\|^2 \le (2+e^{-\frac{\gamma\delta}{2}})\varepsilon$ for $0 \le s \le t < \delta$, which implies that the map $s \in [0,\infty) \mapsto \eta(e^{\gamma s}\cdot) \in L^2$ is continuous. Therefore the X_2-valued Riemann integral $\int_0^t \eta(e^{\gamma s}x)ds$ exists. Since the equation

$$\frac{f(e^{\gamma t}x)-f(x)}{t} - \gamma x f'(x) = \frac{1}{t}\int_0^t \gamma e^{\gamma s} x f'(e^{\gamma s}x) ds - \eta(x)$$

holds, for $0 < t < \delta$, the norm of the third term of (3.2) can be rewritten as follows:

$$\begin{aligned} \left\|\frac{f(e^{\gamma t}x)-f(x)}{t} - \gamma x f'(x)\right\| &= \left\|\frac{1}{t}\int_0^t \eta(e^{\gamma s}x)ds - \eta(x)\right\| \\ &\le \frac{1}{t}\int_0^t \|\eta(e^{\gamma s}x)-\eta(x)\| ds < (2+e^{-\frac{\gamma\delta}{2}})\varepsilon, \end{aligned}$$

where $\int_0^t \eta(e^{\gamma s}x)ds$ is the X_2-valued Riemann integral.

This implies that $\|\frac{f(e^{\gamma t}x)-f(x)}{t} - \gamma x f'(x)\|$ goes to zero as $t \to 0$. So f belongs to $D(A)$. Hence $D(A) = D_2$.

Put $\alpha = \min\{\Re(h(x)) \mid x \in [0,1]\}$ and

$$U = \left\{\lambda \in \mathbb{C} \mid \Re(\lambda) < \alpha - \frac{\gamma}{2}\right\}.$$

Since we assume $\alpha > \frac{\gamma}{2}$, the set U intersects the imaginary axis. For $\lambda \in U$, it is easy to see that $f_\lambda(x) = \exp(-\frac{1}{\gamma}\int_x^1 \frac{\lambda - h(s)}{s}ds)$ belongs to $D_2 = D(A)$ and $Af_\lambda = \lambda f_\lambda$, i.e. f_λ is an eigenvector of A. So U is an open subset of the point spectrum of A.

(iii) For $\phi \in X_2^* = X_2$, we have

$$F_\phi(\lambda) = \langle \phi, f_\lambda\rangle_{L^2} = \int_0^1 \phi(x) f_\lambda(x)dx. \tag{3.3}$$

For $\lambda \in U$, we shall show that $\dfrac{\partial f_\lambda(x)}{\partial \lambda}$ exists. For each $x \in (0,1)$, $f_\lambda(x)$ is differentiable with respect to λ on U and

$$\begin{aligned}|\frac{1}{\nu}\{f_{\lambda+\nu}(x) - f_\lambda(x)\}| &= |\frac{1}{\nu}\left\{e^{-\frac{1}{\gamma}\int_x^1 \frac{\lambda+\nu-h(s)}{s}ds} - e^{-\frac{1}{\gamma}\int_x^1 \frac{\lambda-h(s)}{s}ds}\right\}| \\ &= |e^{-\frac{1}{\gamma}\int_x^1 \frac{\lambda-h(s)}{s}ds}\frac{1}{\nu}\left\{e^{-\frac{1}{\gamma}\int_x^1 \frac{\nu}{s}ds} - 1\right\}| \\ &\cdot \quad x^{\frac{\Re(\lambda)-\alpha}{\gamma}} \cdot \frac{\log x}{\gamma} x^{\frac{\theta\nu}{\gamma}},\end{aligned}$$

with some $0 < \theta < 1$. Since $\frac{\Re(\lambda)-\alpha}{\gamma} > -\frac{1}{2}$, we can choose a small number $\nu_0 > 0$ satisfying $\frac{\Re(\lambda)-\alpha}{\gamma} + \frac{\nu_0}{\gamma} > -\frac{1}{2}$. Furthermore, we can take $b > 0$ satisfying $\frac{\Re(\lambda)-\alpha}{\gamma} + \frac{\nu_0}{\gamma} - b > -\frac{1}{2}$. Since $x^b \log x \in C((0,1],\mathbb{C})$ and $\lim_{x\to 0} x^b \log x = 0$, there exists $M > 0$ such that $||x^b \log x||_\infty \cdot\ M$. Put $\beta = \frac{\Re(\lambda)-\alpha}{\gamma} + \frac{\nu_0}{\gamma} - b$. Then $|\frac{1}{\nu}\{f_{\lambda+\nu}(x) - f_\lambda(x)\}| \cdot\ \frac{M}{|\gamma|}x^\beta$ and the function $\frac{M}{|\gamma|}x^\beta$ belongs to $L^2([0,1],\mathbb{C})$, since $\beta > -\frac{1}{2}$. By putting $\psi(x) = |\phi(x)|\frac{Mx^\beta}{|\gamma|}$, we have $\psi \in L^1([0,1],\mathbb{C})$ and

$$\left|\phi(x)\frac{1}{\nu}\{f_{\lambda+\nu}(x) - f_\lambda(x)\}\right| \cdot\ \psi(x)$$

for any ν with $0 < |\nu| \cdot\ \nu_0$ and $x \in [0,1]$. So we can apply Lebesgue's dominated convergence theorem to the equation (3.3). Hence F_ϕ is analytic.

(iv) In a similar way to (iv) in the proof of Theorem 1, we can show that $\phi = 0$ if $F_\phi(\lambda) = 0$ for all $\lambda \in U$.

By (i) to (iv), if $\min\{\Re(h(x)) \mid x \in [0,1]\} > \frac{\gamma}{2}$ then all assumptions of Theorem A hold. So $\{T_t\}_{t\geq 0}$ is chaotic by Theorem A.

$\square$

We have the following similar corollary to that of Theorem 1.

Corollary. *Let Y_2 be the space $L^2([1,\infty),\mathbb{C})$. We consider the following initial value problem of a partial differential equation:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma x \dfrac{\partial u}{\partial x} + h(x)u \\ u(0,x) = f(x) \end{cases}$$

where $\gamma > 0$, $f \in Y_2$, $h \in C([1,\infty),\mathbb{C})$ and $\lim_{x\to\infty} h(x)$ exists. Then the solution semigroup $\{T_t\}_{t\geq 0}$ ($T_t f(x) = \exp\left\{\int_0^t h(e^{\gamma(t-s)}x)\,ds\right\} f(e^{\gamma t}x)$) to the partial differential equation is a strongly continuous semigroup on Y_2.

Moreover if $\inf\{\Re(h(x)) \mid x \in [1,\infty)\} > \frac{\gamma}{2}$*, then $\{T_t\}_{t\geq 0}$ is chaotic.*

§4. Chaotic semigroups on $C_0(I,\mathbb{C})$ related to chaotic translation semigroups on admissible weighted function spaces

Let I be the interval $[0,\infty)$ and $\widetilde{X}$ be the space $C_0(I,\mathbb{C})$ of all complex-valued continuous functions on I satisfying $\lim_{x\to\infty} f(x) = 0$ with $\|f\|_\infty = \sup_{x\in I}|f(x)|$. We shall consider the following partial differential equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} + h(x)u \\ u(0,x) = f(x), \end{cases} \tag{4.1}$$

where h is a bounded continuous function on I and $f \in \widetilde{X}$.

By using the representation formula $e^{\int_x^{x+t} h(s)ds} f(x+t)$ of the classical solution of (4.1), we define the bounded linear operator $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ on $\widetilde{X}$ as follows:

$$\widetilde{T}_t f(x) = e^{\int_x^{x+t} h(s)ds} f(x+t) \qquad \text{for} \quad f \in \widetilde{X}.$$

According to the paper [1], we call $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ *the solution semigroup* on $\widetilde{X}$ to the partial differential equation (4.1).

If λ is an eigenvalue of the infinitesimal generator A of the strongly continuous semigroup $\left\{\widetilde{T}_t\right\}_{t\geq 0}$, then the eigenfunction f_λ is of the form $f_\lambda(x) = \text{const.} \times e^{\lambda x - \int_0^x h(s)ds}$. It seems impossible that there exists an open subset of the point spectrum of A, which intersects the imaginary axis. So we cannot apply the method of Theorem A to show that $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ is chaotic. Hence we introduce the space $C_{0,\rho}(I,\mathbb{C})$ defined by an admissible weight function ρ.

By an *admissible weight function* on I we mean a measurable function $\rho : I \to \mathbb{R}$ satisfying the following conditions:
(i) $\rho(x) > 0$ for all $x \in I$;
(ii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(x) \cdot \ Me^{\omega t}\rho(t+x)$ for all $x \in I$ and $t > 0$.

For an admissible weight function ρ on $I = [0,\infty)$, we consider the following function space:

$$C_{0,\rho}(I,\mathbb{C}) = \left\{f : I \to \mathbb{C} \mid f \text{ continuous}, \lim_{x\to\infty} \rho(x)f(x) = 0\right\}$$

with $\|f\|_\rho = \sup_{x\in I} |f(x)|\rho(x)$.

Let X be the space $C_{0,\rho}(I,\mathbb{C})$ defined by an admissible weight function ρ. For $t \geq 0$, we define $T_t \in \mathfrak{L}(X)$ by

$$T_t f(x) = f(x+t)$$

for $f \in X$. We call $\{T_t\}_{t\geq 0}$ the *translation semigroup* on X.

Put $\rho(x) = e^{-\int_0^x h(s)ds}$. Since h is a bounded function, there exists a constant $\omega > 0$ such that $h(x) \cdot \ \omega$ for any $x \in I$. So

$$\int_x^{x+t} h(s)ds \cdot \ \omega t$$

holds. Rewriting the inequality we have

$$e^{-\int_0^x h(s)ds} \cdot \ e^{\omega t} \cdot e^{-\int_0^{x+t} h(s)ds}.$$

So ρ is continuous by the continuity of h, and ρ is an admissible weight function since $\rho(x) \cdot \ e^{\omega t}\rho(x+t)$ holds.

By the definition of ρ, the equality $-\dfrac{\rho'(x)}{\rho(x)} = h(x)$ holds. So the partial differential equation (4.1) is rewritten as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} - \dfrac{\rho'(x)}{\rho(x)} u \\ u(0,x) = f(x) \end{cases}$$

with a continuous admissible weight function ρ. Hence we have

$$u(t,x) = \widetilde{T}_t f(x) = \frac{\rho(x)}{\rho(x+t)} f(x+t) \in C_0(I,\mathbb{C}). \tag{4.2}$$

Recall that $\widetilde{X}$ is the space $C_0(I,\mathbb{C})$ of all complex-valued continuous functions on I satisfying $\lim_{x\to\infty} f(x) = 0$ with $\|f\|_\infty = \sup_{x\in I}|f(x)|$. We shall define the following operator $\varphi : X \to \widetilde{X}$ as

$$\varphi(f)(x) = \rho(x)f(x)$$

for $f \in X$ and for $x \in I$.

It is easy to see that the following diagram commutes: $\begin{array}{ccc} X & \xrightarrow{T_t} & X \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ \widetilde{X} & \xrightarrow[\widetilde{T}_t]{} & \widetilde{X} \end{array}$.

Since $\rho(x) > 0$ for all $x \in I$, φ is an isometric isomorphism. So we have the following.

Proposition 3. *Let X be the space $C_{0,\rho}(I,\mathbb{C})$ with a continuous admissible weight function ρ and $\{T_t\}_{t\geq 0}$ be the translation semigroup on X. Let $\widetilde{X}$ be the space $C_0(I,\mathbb{C})$ and $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ be the semigroup defined by (4.2). Then*

(1) *$\{T_t\}_{t\geq 0}$ is hypercyclic on X iff $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ is hypercyclic on $\widetilde{X}$.*

(2) *$\{T_t\}_{t\geq 0}$ is chaotic on X iff $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ is chaotic on $\widetilde{X}$.*

To prove the following Theorem 4, we need the next result.

Theorem B ([7]). *Let ρ be an admissible weight function and X be $C_{0,\rho}(I,\mathbb{C})$ with $I = [0,\infty)$. Then the following assertions are equivalent:*

(i) *the translation semigroup $\{T_t\}_{t\geq 0}$ on X is chaotic;*

(ii) *for any $\varepsilon > 0$ and for any $l > 0$, there exists $P > 0$ such that $\rho(l + nP) < \varepsilon \quad$ for all $n \in \mathbb{N}$.*

Theorem 4. *Let $\widetilde{X} = C_0(I,\mathbb{C})$ with $I = [0,\infty)$. We consider the partial differential equation:*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} + h(x)u \\ u(0,x) = f(x) \quad \text{with some } f \in \widetilde{X}, \end{cases}$$

where h is a bounded continuous function on I.

Then the solution semigroup $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ is a strongly continuous semigroup on $\widetilde{X}$. Moreover if $h(x)$ satisfies $\int_0^\infty h(s)ds = \infty$, then $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ is chaotic.

Proof. By the relation $\widetilde{T}_t f(x) = \dfrac{\rho(x)}{\rho(x+t)} f(x+t)$, it is easy to see that $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ is a semigroup.

To show the strong continuity of $\left\{\widetilde{T}_t\right\}_{t\geq 0}$, we shall show the continuity at $t = 0$. Put $\rho(x) = e^{-\int_0^x h(s)ds}$. Since h is a bounded function, there exists a constant $\omega > 0$ such that $h(x) \leq \omega$ for any $x \in I$. For any $\varepsilon > 0$ there exists $R > 0$ such that $|f(x)| < \dfrac{\varepsilon}{3e^{\omega}}$ for $x > R$. Then $|u(t,x)| = |\dfrac{\rho(x)}{\rho(x+t)} f(x+t)| \leq e^{\omega t}|f(x+t)| \leq \dfrac{\varepsilon}{3}$ for $0 \leq t < 1$ and $x > R$. Since $u(t,x)$ is uniformly continuous on $[0,1] \times [0,R]$, there exists $1 > \delta > 0$ such that $|u(t,x) - u(0,x)| < \dfrac{\varepsilon}{3}$ for $0 \leq t < \delta$ and $x > 0$. So

$$
\begin{aligned}
||\widetilde{T}_t f - f|| &= \sup_{x\in[0,\infty)} |u(t,x) - u(0,x)| \\
&\leq \sup_{x\in[0,R]} |u(t,x) - u(0,x)| + \sup_{x\in[R,\infty)} |u(t,x) - u(0,x)| < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon
\end{aligned}
$$

for $0 \leq t < \delta$. Hence $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ is a strongly continuous semigroup.

We shall check that $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ is chaotic on $C_0(I,\mathbb{C})$. By the assumption $\int_0^\infty h(s)ds = \infty$, we have $\lim_{x\to\infty} \rho(x) = 0$. By Theorem B, the translation semigroup $\{T_t\}_{t\geq 0}$ is chaotic on $C_{0,\rho}(I,\mathbb{C})$ where $T_t f(x) = f(x+t)$. By Proposition 3, $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ is chaotic on $C_0(I,\mathbb{C})$. □

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